A NON-SEPARABLE CHRISTENSEN'S THEOREM AND SET TRI-QUOTIENT MAPS

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Abstract. For every space X let $\mathcal{K}(X)$ be the set of all compact subsets of X. Christensen [6] proved that if X, Y are separable metrizable spaces and $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ is a monotone map such that any $L \in \mathcal{K}(Y)$ is covered by $F(K)$ for some $K \in \mathcal{K}(X)$, then Y is complete provided X is complete. It is well known [3] that this result is not true for non-separable spaces. In this paper we discuss some additional properties of F which guarantee the validity of Christensen's result for more general spaces.

1. Introduction

All spaces in this paper are assumed to be completely regular.

The following characterization of Polish spaces established by J.P. Christensen [6] (see also [18] for another proof) is well known.

Theorem 1.1. [6] *A separable metric space Y is complete iff there exists a Polish space X and a map $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ such that:*

(1) *F is monotone (i.e. if $K, L \in \mathcal{K}(X)$ with $K \subset L$, then $F(K) \subset F(L)$);*
(2) *$F(\mathcal{K}(X))$ is cofinal in $\mathcal{K}(Y)$ (i.e. for each $L \in \mathcal{K}(Y)$ there is $K \in \mathcal{K}(X)$ with $L \subset F(K)$).*

According to Proposition 2.2(b) and Theorem 1.4 below, Theorem 1.1 remains valid if condition (2) is replaced by the weaker one:

$(2)_c$ For any countable $L \in \mathcal{K}(Y)$ there exists $K \in \mathcal{K}(X)$ with $L \subset F(K)$.

Theorem 1.1 is not valid for non-separable X. Indeed, let $\mathbb{Q}$ be rational numbers and X the discrete sum of all compact subsets of $\mathbb{Q}$. Then there exist a map $F : \mathcal{K}(X) \to \mathcal{K}(\mathbb{Q})$ satisfying conditions (1) and (2), see [3]. Our first principal result shows that Theorem 1.1 remains valid for arbitrary metrizable X and Y if F satisfies an extra condition:

Theorem 1.2. *Let X and Y be metrizable spaces and $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ be a map satisfying conditions* (1), $(2)_c$ *and condition* $(3)_c$ *below:*

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(3)$_c$ *If $U \subset X$ and $V \subset Y$ are non-empty open sets such that for each countable compact set $L \subset V$ there is a compact $K \subset U$ with $L \subset F(K)$, then for any open cover $\mathcal{W}$ of U and any point $y \in V$ there exist a finite subfamily $\mathcal{E} \subset \mathcal{W}$ and a neighborhood V_y of y such that each countable compact $K \subset V_y$ is covered by $F(K)$ for some compact $K \subset \bigcup \mathcal{E}$.*

Then Y is completely metrizable and $densY \leq densX$ provided X is completely metrizable.

Any map $F: \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies (3)$_c$ if X and Y are metrizable with X being separable (see Proposition 2.2(b)). So, Theorem 1.2 is a generalization of Christensen's result.

A non-metrizable analog of Theorem 1.1 was established in [8] (see [4] for related results).

Theorem 1.3. [8] *Let X be a Lindelöf Čech-complete space and $F: \mathcal{K}(X) \to \mathcal{K}(Y)$ be a map satisfying conditions* (1), (2). *If Y is a μ-complete q-space, then Y is also Lindelöf and Čech-complete.*

Recall that X is said to be a *μ-space* or *μ-complete* if every closed and bounded set in X is compact. Here, a set $A \subset X$ is *bounded* in X if each continuous real-valued function on X is bounded on A. All paracompact, in particular, Lindelöf spaces, are μ-complete. The notion of a q-space was introduced in [11]: X is a q-space if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has a cluster point in X. Obviously, every first countable, in particular, every metric space is a q-space.

In order to obtain a general version of Theorem 1.2 which implies Theorem 1.3, we introduce a special type of set-valued maps called *set tri-quotient maps* (see Section 2). Recall that tri-quotient maps (single-valued) introduced by Michael [12] are extensively investigated, see [9], [10], [13], [14], [15], [17], [20].

Every map $F: \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfying conditions (1), (2)$_c$ and (3)$_c$ is a monotone set tri-quotient map (see Proposition 2.4). This allows us to derive Theorem 1.2 and Theorem 1.3 from the following one which in turn follows from Theorem 3.3 (recall that sieve-completeness, see [7] and [12], is a more general property than Čech-completeness and both they are equivalent in the class of paracompact spaces).

Theorem 1.4. *Let X be a sieve-complete space and $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ be a monotone set tri-quotient map. If Y is a μ-space, then Y is also seieve-complete and the Lindelöf number $l(Y)$ of Y is $\leq l(X)$.*

In the last section we apply Theorem 3.3 to show that sieve completeness is preserved under linear continuous surjections between function spaces, see Theorem 4.3. We also establish a locally compact version of Theorem 1.2.

2. Set tri-quotient maps

The topology of a space X is denoted by $\mathcal{T}(X)$.

Let $\mathcal{S}(X) \subset 2^X$. A map $F : \mathcal{S}(X) \to 2^Y$ is called *set tri-quotient* if there exists a map $s : \mathcal{T}(X) \to \mathcal{T}(Y)$ such that:

(str1) $s(U) \subset \bigcup\{F(K) : K \in \mathcal{S}(X) \text{ and } K \subset U\}$;
(str2) $s(X) = Y$;
(str3) $U \subset V$ implies $s(U) \subset s(V)$;
(str4) if $y \in s(U)$ and if $\mathcal{W}$ is a cover of $\bigcup\{K \in F^{-1}(y) : K \subset U\}$ by open subsets of X, then $y \in s(\bigcup \mathcal{E})$ for some finite $\mathcal{E} \subset \mathcal{W}$.

In the above definition $F^{-1}(y)$ stands for the family $\{K \in \mathcal{S}(X) : y \in F(K)\}$. Let us also observe that conditions ($str1$) and ($str4$) imply that F is surjective, i.e. $Y = \bigcup\{F(K) : K \in \mathcal{S}(X)\}$.

There is a similarity between set tri-quotient maps and Michael's tri-quotient maps [12]. To clarify this similarity, let us consider another class of maps introduced in [8].

A map $F : X \to 2^Y$ is said to be *generalized tri-quotient* if one can assign to each open $U \subset X$ an open $t(U) \subset Y$ such that:

(gtr1) $t(U) \subset F(U) = \cup\{F(x) : x \in U\}$;
(gtr1) $t(X) = Y$;
(gtr1) $U \subset V$ implies $t(U) \subset t(V)$;
(gtr1) if $y \in t(U)$ and if $\mathcal{W}$ is a cover of $F^{-1}(y) \cap U$ by open subsets of X, then $y \in t(\bigcup \mathcal{E})$ for some finite $\mathcal{E} \subset \mathcal{W}$.

We call the function $t : \mathcal{T}(X) \to \mathcal{T}(Y)$ an *assignment* for F. By (gtr1), every generalized tri-quotient map is surjective, i.e. $Y = F(X)$. When $F : X \to Y$ is single-valued and continuous, the above definition coincides with the definition of a tri-quotient map [12]. It was shown [8, Proposition 2.1] that $F : X \to 2^Y$ is generalized tri-quotient if and only if the projection $\pi_Y : G(F) \to Y$ is tri-quotient, where $G(F)$ is the graph of F. This result, compared with [16, Theorem 2.4], shows that generalized tri-quotient maps (as well as, set tri-quotient maps) are different from the class of set-valued tri-quotient maps introduced by Ostrovsky [16].

Next lemma describes the connection between generalized tri-quotient and set tri-quotient maps.

Lemma 2.1. *Let $F : X \to 2^Y$ be a generalized tri-quotient map. Then $\Phi : 2^X \to 2^Y$, $\Phi(A) = \mathrm{cl}_Y(F(A))$, is monotone set tri-quotient.*

Proof. It follows from the definition that Φ is monotone. Let $t : \mathcal{T}(X) \to \mathcal{T}(Y)$ be an assignment for F. We define $s(U) = t(U)$ for every open $U \subset X$. Obviously, s satisfies the first three conditions (str1)-(str3). Since $F^{-1}(y) \cap U \subset \bigcup\{K \in \Phi^{-1}(y) : K \subset U\}$ for all $y \in Y$ and $U \in \mathcal{T}(X)$, condition (str4) also holds. $\square$

Similarly, every tri-quotient map $f : X \to Y$ generates a monotone set tri-quotient map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $F(K) = f(K)$, $K \in \mathcal{K}(X)$.

Now, let us show that the map F from Theorem 1.1 and Theorem 1.3 is monotone set tri-quotient.

Proposition 2.2. *Suppose $F : \mathcal{K}(X) \to \mathcal{K}(Y)$. Then we have:*

(a) *F is monotone set tri-quotient provided F satisfies conditions* (1) *and* (2)*, X is Lindelöf and Y a μ-complete q-space;*

(b) *F satisfies condition $(3)_c$ provided X is separable metric and Y is first countable. Moreover, F is monotone set tri-quotient if F satisfies conditions* (1) *and* $(2)_c$.

Proof. To prove (a), suppose X is Lindelöf, Y is a μ-complete q-space and F satisfies conditions (1) and (2). We say that a set $A \subset Y$ is F-covered by a set $B \subset X$ if for any compact $L \subset A$ there exists a compact $K \subset B$ with $L \subset F(K)$.

Claim 2.3. *Let $U \subset X$ be functionally open and $V \subset Y$ open such that V is F-covered by U. If $\mathcal{W}$ is an open cover of U and $y \in V$, then there exists a neighborhood V_y of y and a finite subfamily $\mathcal{E} \subset \mathcal{W}$ such that V_y is F-covered by $\bigcup \mathcal{E}$.*

Since U is functionally open, it is Lindelöf. So, we can suppose that $\mathcal{W} = \{W_n : n \geq 1\}$ is countable. Let $\{V_n\}$ be a sequence of neighborhoods of y witnessing that y is a q-point and such that $\mathrm{cl}(V_{n+1}) \subset V_n \subset V$ for all n. Assume the claim is false and for each n choose a compact set $L_n \subset V_n$ which is not covered by any $F(K)$, $K \in \mathcal{K}\big(\bigcup_{i=1}^{i=n} W_i\big)$. Then the set

$$L = \Big(\bigcup_{n=1}^{\infty} L_n\Big) \bigcup \Big(\bigcap_{n=1}^{\infty} V_n\Big)$$

is closed. It is bounded in Y because every infinite subset of L has a cluster point. Hence L is compact (recall that Y is a μ-space). Since $L \subset V$ and V is F-covered by U, there is a compact set $K \subset U$ with $L \subset F(K)$. Then $K \subset \bigcup_{i=1}^{i=m} W_i$ for some m. Consequently, L_m is covered by $F(K)$, which contradicts the choice of L_m. The claim is proved.

Now, for every open $U \subset X$ let $s(U)$ be the set of all $y \in Y$ having a neighborhood in Y which is F-covered by a functionally open subset W of X with $W \subset U$. Obviously, $s(U)$ is open in Y (possibly empty) and s satisfies first three conditions from the definition of a set tri-quotient map. To check the last one, let $z \in s(U)$ and $\mathcal{W}$ be a cover of $\bigcup\{K \in F^{-1}(z) : K \subset U\}$ consisting of open in X sets. Then there is a functionally open subset W_0 of X with $W_0 \subset U$ and a neighborhood V_0 of z such that V_0 is F-covered by W_0. Since F is monotone, $U = \bigcup\{K \in F^{-1}(z) : K \subset U\}$, so $\mathcal{W}$ is an open cover of

U. Taking a refinement of $\mathcal{W}$, if necessary, we can assume that each element of $\mathcal{W}$ is functionally open in X. Then $\mathcal{W}_0 = \{G \cap W_0 : G \in \mathcal{W}\}$ is a functionally open cover of W_0. According to Claim 2.3, there exist a neighborhood V_z of z and finite $\mathcal{E}_0 \subset \mathcal{W}_0$ such that V_z is F-covered by $\bigcup \mathcal{E}_0$.

To finish the proof of (a), let $\mathcal{E} = \{G \in \mathcal{W} : G \cap W_0 \in \mathcal{E}_0\}$. Because V_z is F-covered by $\bigcup \mathcal{E}_0$ which is functionally open in X (as a finite union of functionally open sets) and $\bigcup \mathcal{E}_0 \subset \bigcup \mathcal{E}$, we have that $z \in s(\bigcup \mathcal{E})$. Therefore, F is set tri-quotient and monotone.

To prove (b), assume F does not satisfy $(3)_c$. Then there are open sets $U \subset X$ and $V \subset Y$, an open cover $\mathcal{W}$ of U and a point $y \in V$ such that every countable compact set $L \subset V$ is covered by $F(K)$ for some compact set $K \subset U$, but y does not have a neighborhood which is contained in any $\bigcup\{F(K) : K \in \mathcal{K}(\bigcup \mathcal{E})\}$ with $\mathcal{E} \subset \mathcal{W}$ being finite. Since X is separable, we can suppose $\mathcal{W} = \{W_n\}_{n\geq 1}$ is countable. Next, choose neighborhoods $V_n \subset V$ of Y and countable compact sets $L_n \subset V_n$ such that $\{V_n\}_{n\geq 1}$ is a local base at y and L_n is not covered by any $F(K)$, $K \subset \mathcal{K}\big(\bigcup_{i=1}^{i=n} W_i\big)$. Since $L = \big(\bigcup_{n=1}^{\infty} L_n\big) \cup \{y\}$ is countable and compact, there exists a compact set $K \subset U$ with $L \subset F(K)$. As in the proof of Claim 2.3, this contradicts the choice of the sets L_n. Hence, F satisfies condition $(3)_c$.

It follows from Proposition 2.4 below that F is monotone set tri-quotient provided it satisfies conditions (1) and $(2)_c$. □

Proposition 2.4. *Let X and Y be arbitrary spaces. Then any map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfying conditions* (1), $(2)_c$ *and* $(3)_c$ *is monotone set tri-quotient.*

Proof. Because F satisfies (1), it is monotone. For every open $U \subset X$ we define $s(U)$ to be the set of all $y \in Y$ having a neighborhood V_y in Y such that any countable compact $L \subset V_y$ is covered by $F(K)$ for some compact set $K \subset U$. Obviously, $s(U)$ is open in Y. Since F satisfies conditions (1), $(2)_c$ and $(3)_c$, it is easily seen that s satisfies conditions $(str1) - (str4)$. So, F is set tri-quotient. □

3. Sieve-complete spaces

3.1. **Proof of Theorem 1.4.** First, let us recall the definition of a sieve and a sieve-complete space (see [7] and [12]). A *sieve* on a space X is a sequence of open covers $\{U_\alpha : \alpha \in A_n\}_{n\in\mathbb{N}}$ of X, together with maps $\pi_n : A_{n+1} \to A_n$ such that $U_\alpha = \bigcup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for all n and $\alpha \in A_n$. A π-*chain* for such a sieve is a sequence (α_n) such that $\alpha_n \in A_n$ and $\pi(\alpha_{n+1}) = \alpha_n$ for all n. The sieve is *complete* if for every π-chain (α_n), every filter base $\mathcal{F}$ on X which meshes with $\{U_{\alpha_n} : n \in \mathbb{N}\}$ (i.e. every $B \in \mathcal{F}$ meets every U_{α_n}) has a cluster point in X, or equivalently, every filter base $\mathcal{F}$ on X such that each U_{α_n} contains some $P \in \mathcal{F}$ clusters in X. A space X with a complete sieve is called *sieve-complete*.

A sieve $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is said to be *finitely additive* [12] if every cover $\{U_\alpha : \alpha \in A_n\}$, as well as every collection of the form $\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ with $\alpha \in A_n$, is closed under finite unions. When $\mathrm{cl}_X(U_\beta) \subset U_\alpha$ for all $\alpha \in A_n$ and $\beta \in \pi_n^{-1}(\alpha)$, the sieve is called a *strong sieve* [7]. Every sieve-complete space has a finitely additive complete sieve [12, Lemma 2.3], as well as a strong complete sieve [12, Lemma 2.4]. Moreover, the proof of [12, Lemma 2.3] shows that the complete finitely additive sieve which is obtained from a strong complete sieve is also strong. Therefore, every sieve complete space has a strong complete finitely additive sieve.

Let $\mathcal{S}(X) \subset 2^X$. We will use τ_V^+ to denote the *upper Vietoris topology* on $\mathcal{S}(X)$ generated by all collections of the form $\hat{U} = \{H \in \mathcal{S}(X) : H \subset U\}$, where U runs over the open subsets of X.

Lemma 3.1. *If $(\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is finitely additive and a strong complete sieve on X, then $(\{\hat{U}_\alpha : \alpha \in A_n\}, \pi_n)$ is a complete sieve on $(\mathcal{K}(X), \tau_V^+)$.*

Proof. Because $\gamma = (\{U_\alpha : \alpha \in A_n\}, \pi_n)$ is a finitely additive sieve on X, $\hat{\gamma} = (\{\hat{U}_\alpha : \alpha \in A_n\}, \pi_n)$ is a sieve on $(\mathcal{K}(X), \tau_V^+)$. Let us show that $\hat{\gamma}$ is complete. Suppose (α_n) is a π-chain and $\mathcal{F}$ a filter base on $\mathcal{K}(X)$ which meshes with $\{\hat{U}_{\alpha_n}\}$. By [12, Lemma 2.5], $K = \bigcap U_{\alpha_n}$ is a nonempty compact subset of X such that every open $W \supset K$ contains some U_{α_n}. Then every neighborhood $\hat{W}$ of K in $(\mathcal{K}(X), \tau_V^+)$ contains some $\hat{U}_{\alpha_n}$, hence $\hat{W}$ meets every $H \in \mathcal{F}$. Therefore K belongs to the closure (in $(\mathcal{K}(X), \tau_V^+)$) of each $H \in \mathcal{F}$, i.e. K is a cluster point of $\mathcal{F}$ in $(\mathcal{K}(X), \tau_V^+)$. $\square$

The following analogue of q-spaces was introduced in [19] : call X a *wq-space* if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each n, then $\{x_n\}$ is bounded in X. The wq-space property is weaker than q-space property and they are equivalent for μ-spaces.

We say that a set-valued map $F : X \to 2^Y$ is a *wq-map* if every $x \in X$ has a sequence $\{U_n\}$ of neighborhoods such that if $x_n \in U_n$ for each n, then $\bigcup\{F(x_n) : n \in \mathbb{N}\}$ has a compact closure in Y. A version of next lemma was established first in [8, Lemma 2.3]. In the present form it appears in [19, Proposition 3.14], and later on in [5, Theorem 2.2].

Lemma 3.2. [19] *Let $F : X \to 2^Y$ be a wq-map with Y being a μ-space. Then there exists an usco map $\Phi : X \to Y$ such that $F(x) \subset \Phi(x)$ for every $x \in X$.*

Next theorem provides the proof of Theorem 1.4.

Theorem 3.3. *Let X be a sieve-complete space and Y a μ-space. If there exists a monotone set tri-quotient map $F : \mathcal{K}(X) \to 2^Y$ such that each $F(K)$, $K \in \mathcal{K}(X)$, has a compact closure in Y, then Y is sieve-complete and $l(Y) \leq l(X)$.*

Proof. As we already mentioned, there exists a strong complete sieve $\gamma = (\{U_\alpha : \alpha \in A_n\}, \pi_n)$ on X which is finitely additive. Then, according to Lemma 3.1, $\hat{\gamma}$ is a complete sieve on $(\mathcal{K}(X), \tau_V^+)$.

First, let us show that F, considered as a set-valued map from $(\mathcal{K}(X), \tau_V^+)$ into Y, is a wq-map. Since γ is a finitely additive and strong sieve on X, for every $K \in \mathcal{K}(X)$ there is a chain (α_n) such that $K \subset U_{\alpha_n}$ for all n. This yields (see [12, Lemma 2.5]) that $C = \bigcap U_{\alpha_n}$ is compact and $\{U_{\alpha_n}\}$ is a base for C. We assign to K the sequence $\{\hat{U}_{\alpha_n}\}$. If $K_n \in \hat{U}_{\alpha_n}$ for all n, then $H = (\bigcup K_n) \cup C$ is a compact subset of X and, since F is monotone, $\bigcup F(K_n) \subset F(H)$. So, $\bigcup F(K_n)$ has a compact closure in Y. Therefore F is a wq-map and, by Lemma 3.2, there exists an usco map $\Phi : (\mathcal{K}(X), \tau_V^+) \to Y$ with $F(K) \subset \Phi(K)$ for every $K \in \mathcal{K}(X)$. Let us observe that Φ is onto, i.e. $Y = \bigcup\{\Phi(K) : K \in \mathcal{K}(X)\}$. Since the Lindelöf number of $(\mathcal{K}(X), \tau_V^+)$ is $\leq l(X)$, the last equality yields $l(Y) \leq l(X)$.

Because F is set tri-quotient, there is a map $s : \mathcal{T}(X) \to \mathcal{T}(Y)$ satisfying conditions (str1)-(str4). Let $W_\alpha = s(U_\alpha)$ for every n and $\alpha \in A_n$. We are going to show that $\lambda = (\{W_\alpha : \alpha \in A_n\}, \pi_n)$ is a complete sieve on Y. Since all $\gamma_n = \{U_\alpha : \alpha \in A_n\}$ are open covers of X, it follows from conditions (str2) and (str4) that each $y \in Y$ is contained in $s(\bigcup \omega_n)$ for some finite $\omega_n \subset \gamma_n$. But each γ_n is finitely additive, so all systems $\{W_\alpha : \alpha \in A_n\}$, $n \geq 1$, are covers of Y. Similarly, we can show that $W_\alpha \subset \bigcup\{W_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for every n and $\alpha \in A_n$. The inclusions $\bigcup\{W_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subset W_\alpha$ follow from (str3) and $U_\alpha = \bigcup\{U_\beta : \beta \in \pi_n^{-1}(\alpha)\}$. Therefore, λ is a sieve on Y. To show that that λ is a complete sieve, suppose (α_n) is a π-chain and $\mathcal{F}$ is a filter base on Y which meshes with $\{W_{\alpha_n} : n \in \mathbb{N}\}$. Then $\Phi^{-1}(\mathcal{F}) = \{\Phi^{-1}(P) : P \in \mathcal{F}\}$ is a filter base on $(\mathcal{K}(X), \tau_V^+)$.

Claim 3.4. $\Phi^{-1}(\mathcal{F})$ *meshes with* $\{\hat{U}_{\alpha_n} : n \in \mathbb{N}\}$.

If $y \in P \cap W_{\alpha_n}$ for some $P \in \mathcal{F}$ and $n \in \mathbb{N}$, then, by (str1), there is $K \in \mathcal{K}(X)$ with $K \subset U_{\alpha_n}$ and $y \in F(K) \subset \Phi(K)$. Therefore, $K \in \Phi^{-1}(P) \cap \hat{U}_{\alpha_n}$ which completes the proof of the claim.

Since $\hat{\gamma}$ is a complete sieve, $\Phi^{-1}(\mathcal{F})$ has a cluster point, say K_0, in $(\mathcal{K}(X), \tau_V^+)$.

Claim 3.5. $\Phi(K_0) \cap \mathrm{cl}_Y(P) \neq \varnothing$ *for each* $P \in \mathcal{F}$.

Suppose $\Phi(K_0) \cap \mathrm{cl}_Y(P) = \varnothing$ for some $P \in \mathcal{F}$. Let $V \subset Y$ be open, disjoint with P and containing $\Phi(K_0)$. Because Φ is usc, there is a neighborhood $\hat{U}$ of K_0 in $(\mathcal{K}(X), \tau_V^+)$ such that $\Phi(K) \subset V$ for every $K \in \hat{U}$. Since $\hat{U}$ meets $\Phi^{-1}(P)$, $\Phi(K) \subset V$ for some $K \in \Phi^{-1}(P)$ which is a contradiction.

By Claim 3.5, $\mathcal{F}_0 = \{\Phi(K_0) \cap \mathrm{cl}_Y(P) : P \in \mathcal{F}\}$ is a filter base on $\Phi(K_0)$. Because $\Phi(K_0)$ is compact, $\mathcal{F}_0$ has a cluster point. So, $\mathcal{F}$ has a cluster point in Y and λ is a complete sieve on Y. $\square$

Let us observe that the restriction in Theorem 3.3 Y to be μ-complete and F to be monotone were used only to apply Lemma 3.2 in order to find an usco map $\Phi : (\mathcal{K}(X), \tau_V^+) \to \mathcal{K}(Y)$ with $F(K) \subset \Phi(K)$, $K \in \mathcal{K}(X)$. Therefore, the following statement holds:

Corollary 3.6. *Let $F : (\mathcal{K}(X), \tau_V^+) \to \mathcal{K}(Y)$ be usc and set tri-quotient with X a sieve-complete space. Then Y is also sieve-complete.*

Corollary 3.7. *For a μ-space Y the following are equivalent:*

(a) *Y is sieve-complete.*

(b) *There exists a paracompact Čech-complete space X and an open (not necessary continuous) surjection $f : X \to Y$ such that $f(K)$ has a compact closure in Y for every $K \in \mathcal{K}(X)$.*

(c) *There exists a paracompact Čech-complete space X and a monotone set tri-quotient map $F : \mathcal{K}(X) \to \mathcal{K}(Y)$.*

Proof. (a) $\Rightarrow$ (b). This implication follows from [7, Theorem 3.7] stating that every sieve-complete space is an open and continuous image of a paracompact Čech-complete space.

(b) $\Rightarrow$ (c). If f satisfies (b), we simply define $F : \mathcal{K}(X) \to \mathcal{K}(Y)$ by $F(K) = \mathrm{cl}_Y f(K)$. Since f is open, F is set tri-quotient.

(c) $\Rightarrow$ (a). This implication follows from Theorem 3.3. $\square$

3.2. **Proof of Theorem 1.2.** According to Proposition 2.4, Theorem 3.3 and the fact that sieve and Čech-completeness are equivalent in the realm of paracompact spaces, it follows that Y is complete. Moreover, Theorem 3.3 also implies that $densY \leq densX$.

4. Remarks and some applications

Let us consider the following analogs of condition $(3)_c$ in Theorem 1.2:

(3) If $U \subset X$ and $V \subset Y$ are non-empty open sets such that for each compact $L \subset V$ there is a compact $K \subset U$ with $L \subset F(K)$, then for any open cover $\mathcal{W}$ of U and any point $y \in V$ there exists a finite subfamily $\mathcal{E} \subset \mathcal{W}$ and a neighborhood V_y of y such that for each compact $L \subset V_y$ there is a compact $K \subset \bigcup \mathcal{E}$ with $L \subset F(K)$.

(3′) For each open cover $\mathcal{W}$ of X and for each point $y \in Y$ there exists a finite subfamily $\mathcal{E} \subset \mathcal{W}$ and a neighborhood V_y such that every compact $L \subset V_y$ is covered by $F(K)$ for some compact $K \subset \bigcup \mathcal{E}$.

Obviously, conditions (3) and $(3)_c$ are not comparable, while conditions (2) and (3) imply (3′). As in Lemma 2.2(b), one can show that any map $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies condition (3) if X is second countable and Y first countable. Moreover, we have the following lemma whose proof is similar to that one of Proposition 2.4.

Lemma 4.1. *If X and Y are arbitrary spaces and $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfies conditions* (1), (2) *and* (3)*, then F is monotone set tri-quotient.*

We do not know whether Theorem 1.2 is valid when F satisfies conditions (1), (2) and (3′). It seems now that the related claim in [3, Theorem 5.2] was overoptimistic.

It is interesting that a locally compact version of Theorem 1.2 is true if F satisfies conditions (1) and (3′).

Proposition 4.2. *Let X be a locally compact space and $F\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfy conditions* (1) *and* (3′)*. Then Y is also locally compact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$ be an open cover of X such that each U_α has a compact closure in X. Since F satisfies condition (3′), for every $y \in Y$ there exists a neighborhood V_y and a finite $\mathcal{E}_y \subset \mathcal{U}$ such that every compact set $L \subset V_y$ is covered by $F(K)$ for some compact $K \subset \bigcup \mathcal{E}_y$. So, $V_y \subset \bigcup\{F(K) : K \in \mathcal{K}(U_y)\}$, where $U_y = \bigcup\{U : U \in \mathcal{E}_y\}$. Because the closure $\overline{U}_y$ is compact and F is monotone, $\bigcup\{F(K) : K \in \mathcal{K}(U_y)\} \subset F\big(\overline{U}_y\big)$. Hence, each V_y has a compact closure in Y. □

As we already observed, if X is second countable and Y first countable, then condition (2) implies condition (3′). In this case, Proposition 4.2 is valid whenever F satisfies conditions (1) and (2). The example provided in the introduction shows that conditions (1) and (2) are not enough for the validity of Proposition 4.2 if X is not separable.

We are going now to apply Theorem 3.3 for obtaining alternative proofs and improvements of some results from [3] and [19] concerning preservation of Čech completeness under linear surjections between function spaces. Everywhere below $C(X, E)$ denotes the set of all continuous maps from X into E (we write $C_p(X)$ when consider real-valued functions). The set $C(X, E)$ endowed with the compact-open or the pointwise convergence topology is denoted by $C_k(X, E)$ or $C_p(X, E)$, respectively. If $u\colon C_k(X, E) \to C_p(Y, F)$ is a linear map, where E and F are normed spaces, then for every $y \in Y$ there exists a continuous linear map $\mu_y\colon C_k(X, E) \to F$ defined by $\mu_y(f) = u(f)(y)$, $f \in C_k(X, E)$. Following Arhangel'skii [1], we define the support $supp\big(\mu_y\big)$ of μ_y to be the set of all $x \in X$ such that for every neighborhood U of x in X there is $f \in C(X, E)$ with $f(X \backslash U) = 0$ and $\mu_y(f) \neq 0$, see [19]. So, we can consider the set-valued map $\varphi\colon Y \to 2^X$, $\varphi(y) = supp\big(\mu_y\big)$. This map has the following properties (see [2], [19]):

(a) φ is lower semi-continuous;
(b) if L is a bounded set in Y, then $\varphi(L)$ is bounded in X;
(c) if K is a bounded set in X, then the set $\varphi^*(K) = \{y \in Y : \varphi(y) \subset K\}$ is bounded in Y;

(d) if u is surjective, then $\varphi(y) \neq \varnothing$ for all $y \in Y$.

It is shown in [3, Theorem 3.3] that if $u\colon C_p(X) \to C_p(Y)$ is a continuous linear surjection with X and Y metrizable, then Y is Čech-complete provided so is X. This result was generalized in [19, Corollary 3.15] to the case of non-metrizable X and Y and function spaces of maps into normed spaces (see the hypotheses of Theorem 4.3 below). Under the same hypotheses, we can establish a sieve completeness version of this result. Of course, if X and Y are paracompact spaces, then Theorem 4.3 and [19, Corollary 3.15] are equivalent. In such a situation, Theorem 4.3 provides an alternative proof of [19, Corollary 3.15].

Theorem 4.3. *Let $u\colon C_k(X,E) \to C_p(Y,F)$ be a continuous linear surjection, where both X and Y are μ-spaces and Y a wq-space. If X is sieve-complete, then so is Y.*

Proof. Since X is a μ-space, Y is a wq-space and φ satisfies condition (b), φ is a wq-map. So, by Lemma 3.2, there exists an usco map $\phi\colon Y \to 2^X$ such that $\varphi(y) \subset \phi(y)$ for every $y \in Y$. Now, define the map $F\colon \mathcal{K}(X) \to 2^Y$ by $F(K) = \phi^*(K)$. Let us note that $F(K)$ may not be a compact subset of Y, but it has a compact closure in Y. Indeed, $F(K) \subset \varphi^*(K)$ and the μ-completeness of Y implies that the set $\varphi^*(K)$ is compact as a closed and bounded subset of Y (it is closed because φ is lower semi-continuous, and it is bounded because of (c)). For every open $U \subset$ let $s(U) = \phi^*(U)$. Since ϕ is upper semi-continuous, every $s(U)$ is open in Y. We are going to show that s satisfies conditions $(str1)-(str4)$. Because $\varphi(y) \neq \varnothing$ for all $y \in Y$, the sets $\phi(y)$, $y \in Y$, are non-empty and compact. This yields that s satisfies conditions $(str1)$ and $(str2)$. Obviously, condition $(str3)$ also holds. Finally, if $y \in s(U)$ and $\mathcal{W}$ is an open cover of U, then $\phi(y) \subset U$ and choose a finite family $\mathcal{E} \subset \mathcal{W}$ covering $\phi(y)$. So, $y \in s\big(\bigcup \mathcal{E}\big)$. Therefore, F is set tri-quotient and we can apply Theorem 3.3 to conclude that Y is sieve-complete. □

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